

ZERO DIFFUSION-DISPERSION LIMITS FOR SCALAR CONSERVATION LAWS

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Abstract. We consider solutions of hyperbolic conservation laws regularized with vanishing diffusion and dispersion terms. Following a pioneering work by Schonbek, we establish the convergence of the regularized solutions toward discontinuous solutions of the hyperbolic conservation law. The proof relies on the method of compensated compactness in the L^2 setting. Our result improves upon Schonbek's earlier results and provides an optimal condition on the balance between the relative sizes of the diffusion and the dispersion parameters. A convergence result is also established for multi-dimensional conservation laws by relying on DiPerna's uniqueness theorem for entropy measure-valued solutions.

Key words and phrases: conservation law, shock wave, entropy solution, measure-valued solution, diffusion, dispersion, singular limit, a priori estimate.

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1. INTRODUCTION

We study here the convergence of solutions of the partial differential equation ($\epsilon \rightarrow 0+$, $\delta = \delta(\epsilon) \rightarrow 0$)

$$u_t + f(u)_x = \epsilon u_{xx} + \delta u_{xxx}, \quad u = u^\epsilon(x, t), \quad x \in \mathbf{R}, \quad t \geq 0, \quad (1.1)$$

toward weak solutions of the corresponding hyperbolic conservation laws:

$$u_t + f(u)_x = 0, \quad u = u(x, t), \quad x \in \mathbf{R}, \quad t \geq 0, \quad (1.2)$$

where the flux $f : \mathbf{R} \rightarrow \mathbf{R}$ is a smooth function with (at most) linear growth at infinity, that is, for some $M > 0$

$$|f'(u)| \leq M, \quad u \in \mathbf{R}.$$

Equations of the form (1.1)-(1.2) arise in fluid dynamics when both viscosity (diffusion) and capillarity (dispersion) play a role. The diffusion ϵ smoothes out the discontinuous solutions of (1.2), while the dispersion δ causes high-frequency oscillations.

In this paper, we establish that the solutions u^ϵ of (1.1) converge toward a weak solution of (1.2) provided

$$\delta = O(\epsilon^2). \quad (1.3)$$

When the stronger condition

$$\delta = o(\epsilon^2) \quad (1.4)$$

holds, we prove that the limit coincides with the entropy solution determined by Kruzkov's theory [10]. We point out that these conditions are sharp since, in the limiting case,

$$\delta = K \epsilon^2 \quad \text{for some } K \in \mathbf{R}, \quad (1.5)$$

limiting solutions may violate Kruzkov's entropy conditions [8, 5, 13, 2]. Furthermore, when (1.3) is violated, the solutions are highly-oscillatory and fail to converge in any strong topology as noted by Lax and Levermore [12]. (See also Lax [11].)

The singular limit problem above was first tackled by Schonbek [19], who established the optimal rate (1.3) for Burgers equation, that is,

$$f(u) = \frac{u^2}{2},$$

and for the class of flux-functions

$$f(u) = \frac{u^{2p+1}}{2p+1}, \quad p \geq 1.$$

She also gave a convergence result for general fluxes with quadratic growth at infinity, however under the stronger condition on $\delta = O(\epsilon^3)$. As another important contribution in [19], Schonbek introduces a generalization of the method of compensated compactness (Tartar [21] and Murat [18]) allowing to handle sequences that are bounded in L^p for finite $p > 1$ only. Next, following [19], LeFloch and Natalini [15] studied equations like (1.1) but with nonlinear (even singular) diffusion, and

established strong convergence results toward entropy solutions of (1.2). See also a convergence result for systems in Hayes and LeFloch [6].

In the second part of this paper, we also deal with the convergence of solutions of multi-dimensional equations similar to (1.1)-(1.2). For multi-dimensional equations, the compensated compactness method no longer applies and the proofs are based instead on DiPerna's uniqueness theory for entropy measure-valued solutions (DiPerna [4], Szepessy [20], and Kondo and LeFloch [9]). Our approach is similar to Correia and LeFloch [3] where nonlinear diffusion terms are treated under a strong assumption on the ratio of the dispersion to the diffusion.

To summarize, the main contribution in the present paper is the derivation of a priori estimates (Theorems 2.1 and 3.1) which cover general flux-functions (with at most linear growth at infinity) and lead to an optimal condition on the balance between the diffusion and the dispersion (Theorems 2.2 and 3.2).

Further material on classical and nonclassical entropy solutions generated by diffusive-dispersive limits can be found in [1, 2, 5, 6, 7, 8, 13, 14, 15, 16, 17, 19]

2. ONE-DIMENSIONAL CONSERVATION LAWS

Consider a family u^ϵ of smooth solutions to

$$u_t + f(u)_x = \epsilon u_{xx} + \delta u_{xxx}, \quad u = u^\epsilon(x, t), \quad (2.1)$$

$$u(x, 0) = u_0^\epsilon(x), \quad x \in \mathbf{R}, \quad (2.2)$$

where $\epsilon \rightarrow 0+$ and $\delta = \delta(\epsilon) \rightarrow 0$. Under suitable conditions on the initial data $u_0^\epsilon : \mathbf{R} \rightarrow \mathbf{R}$, the solutions (and their derivatives) decay rapidly at infinity, so that all the a priori estimates given below are rigorously justified. We want to show that the solution of (2.1)-(2.2) converges toward a weak solution of the problem

$$u_t + f(u)_x = 0, \quad u = u^\epsilon(x, t), \quad (2.3)$$

$$u(x, 0) = u_0(x), \quad x \in \mathbf{R}, \quad (2.4)$$

where $u_0 : \mathbf{R} \rightarrow \mathbf{R}$ is a given initial data. A minimum requirement is the weak convergence (for instance in $L^2(\mathbf{R})$)

$$u_0^\epsilon \rightharpoonup u_0,$$

which is always assumed throughout this paper. The following convergence theorem covers both cases where the diffusion are in balance or dominates the dispersion.

Theorem 2.1. *Suppose that the flux-function f is Lipschitz continuous on \mathbf{R} and that the initial data u_0 belong to $L^2(\mathbf{R})$. Then the solution u^ϵ of (2.1)-(2.2) satisfies the following a priori estimates:*

$$\|u^\epsilon(t)\|_{L^2(\mathbf{R})} \leq \|u_0^\epsilon\|_{L^2(\mathbf{R})}, \quad t \geq 0, \quad (2.5a)$$

$$\sqrt{2\epsilon} \|u_x^\epsilon\|_{L^1(\mathbf{R}^+, L^2(\mathbf{R}))} \leq \|u_0^\epsilon\|_{L^2(\mathbf{R})}, \quad (2.5b)$$

$$\sqrt{\delta} \|u_x^\epsilon(t)\|_{L^2(\mathbf{R})} \leq \sqrt{2\|f'\|_\infty} \|u_0^\epsilon\|_{L^2(\mathbf{R})} + \sqrt{\delta} \|u_{0x}^\epsilon\|_{L^2(\mathbf{R})}, \quad t \geq 0, \quad (2.5c)$$

and

$$\sqrt{\epsilon \delta} \|u_{xx}\|_{L^1(\mathbf{R}^+, L^2(\mathbf{R}))} \leq \sqrt{2 \|f'\|_\infty} \|u_0^\epsilon\|_{L^2(\mathbf{R})} + \sqrt{\delta} \|u_{0x}^\epsilon\|_{L^2(\mathbf{R})}. \quad (2.5d)$$

Proof. Throughout the calculation and for simplicity, we omit the upper-index ϵ . To any smooth function $U : \mathbf{R} \rightarrow \mathbf{R}$ we can associate a “flux” $F : \mathbf{R} \rightarrow \mathbf{R}$ by $F'(u) = U'(u)f'(u)$, $u \in \mathbf{R}$. Multiplying (2.1) by $U'(u)$ we find

$$U(u)_t + F(u)_x = \epsilon (U'(u)u_x)_x - \epsilon U''(u)u_x^2 + \delta (U'(u)u_{xx})_x - \delta U''(u)u_x u_{xx}.$$

Integrating over the whole space, it follows that

$$\begin{aligned} \frac{d}{dt} \int_{\mathbf{R}} U(u) dx + \epsilon \int_{\mathbf{R}} U''(u) u_x^2 dx &= \delta \int_{\mathbf{R}} U''(u) \left(\frac{u_x^2}{2} \right)_x dx \\ &= -\frac{\delta}{2} \int_{\mathbf{R}} U'''(u) u_x^3 dx. \end{aligned} \quad (2.6)$$

Integrating in time over some interval $(0, t)$, we arrive at the general identity:

$$\int_{\mathbf{R}} U(u(t)) dx + \epsilon \int_0^t \int_{\mathbf{R}} U''(u) u_x^2 dx dt = \int_{\mathbf{R}} U(u_0) dx - \frac{\delta}{2} \int_0^t \int_{\mathbf{R}} U'''(u) u_x^3 dx dt. \quad (2.7)$$

Choosing first $U(u) = u^2$ in (2.7), we see that

$$\int_{\mathbf{R}} u(t)^2 dx + 2\epsilon \int_0^t \int_{\mathbf{R}} u_x^2 dx = \int_{\mathbf{R}} u_0^2(x), \quad (2.8)$$

which gives immediately (2.5a) and (2.5b).

Next, we differentiate (2.1) with respect to x and we multiply by u_x :

$$\frac{1}{2} (u_x^2)_t + u_x (f'(u) u_x)_x = \epsilon (u_x u_{xx})_x - \epsilon u_{xx}^2 + \delta (u_x u_{xxx} - \frac{1}{2} u_{xx}^2)_x.$$

Integrating in space, we get

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbf{R}} u_x^2 dx + \epsilon \int_{\mathbf{R}} u_{xx}^2 dx = \int_{\mathbf{R}} u_{xx} f'(u) u_x dx = -\frac{1}{2} \int_{\mathbf{R}} f''(u) u_x^3 dx.$$

Hence, integrating over some interval $(0, t)$, we find

$$\int_{\mathbf{R}} u_x(t)^2 dx + 2\epsilon \int_0^t \int_{\mathbf{R}} u_{xx}^2 dx dt = \int_{\mathbf{R}} u_{0x}^2 dx - \int_0^t \int_{\mathbf{R}} f''(u) u_x^3 dx dt. \quad (2.9)$$

Multiply (2.9) by δ and add it up with (2.7):

$$\begin{aligned} &\delta \int_{\mathbf{R}} u_x(t)^2 dx + 2\epsilon \delta \int_0^t \int_{\mathbf{R}} u_{xx}^2 dx dt \\ &= \int_{\mathbf{R}} U(u_0) dx - \int_{\mathbf{R}} U(u(t)) dx + \delta \int_{\mathbf{R}} u_{0x}^2 dx - \epsilon \int_0^t \int_{\mathbf{R}} U''(u) u_x^2 dx dt \\ &\quad - \delta \int_0^t \int_{\mathbf{R}} f''(u) u_x^3 dx dt - \frac{\delta}{2} \int_0^t \int_{\mathbf{R}} U'''(u) u_x^3 dx dt. \end{aligned}$$

Choosing U given by

$$U(u) = -2 \int_0^u (f(v) - f(0)) dv \quad (2.10)$$

the last two terms in the above identity cancel out. Since

$$-c \leq \frac{U''(u)}{2} \leq c := \|f'\|_\infty, \quad \text{for all } u \in \mathbf{R},$$

thus

$$-c u^2 \leq U(u) \leq c u^2, \quad \text{for all } u \in \mathbf{R},$$

we finally obtain

$$\begin{aligned} & \delta \int_{\mathbf{R}} u_x(t)^2 dx + 2\epsilon \delta \int_0^t \int_{\mathbf{R}} u_{xx}^2 dx dt \\ & \leq \int_{\mathbf{R}} c u_0^2 dx + \int_{\mathbf{R}} c u(t)^2 dx + \delta \int_{\mathbf{R}} u_{0x}^2 dx + 2\epsilon \int_0^t \int_{\mathbf{R}} c u_x^2 dx dt. \end{aligned}$$

Hence using (2.8)

$$\delta \int_{\mathbf{R}} u_x(t)^2 dx + 2\epsilon \delta \int_0^t \int_{\mathbf{R}} u_{xx}^2 dx dt \leq 2c \int_{\mathbf{R}} u_0^2 dx + \delta \int_{\mathbf{R}} u_{0x}^2 dx,$$

which leads to (2.5c) and (2.5d). The proof of Theorem 2.1 is completed. \square

Recall that by Kruzkov' theory, given $u_0 \in L^2(\mathbf{R})$ the Cauchy problem (2.3)-(2.4) admits a unique entropy solution $u \in L^\infty(\mathbf{R}_+, L^2(\mathbf{R}))$ in the sense of Kruzkov's theory. See [10, 4, 20, 9].

Theorem 2.2. *Assume that, for some constant $C_0 > 0$ independent of ϵ ,*

$$\|u_0^\epsilon\|_{L^2(\mathbf{R})} + \sqrt{\delta} \|u_{0x}^\epsilon\|_{L^2(\mathbf{R})} \leq C_0. \quad (2.11)$$

- (1) *As $\epsilon \rightarrow 0$ with $\delta = O(\epsilon^2)$ (a subsequence of) the solution u^ϵ of (2.1)-(2.2) converges in $L_{\text{loc}}^p(\mathbf{R}_+, L_{\text{loc}}^q(\mathbf{R}))$ (for all $1 < p < \infty$ and $1 < q < 2$) toward a weak solution of the problem (2.3)-(2.4).*
- (2) *If the stronger condition $\delta = o(\epsilon^2)$ holds, then the limit is the unique entropy solution in the sense of Kruzkov.*

In Case (1) a subsequence of u^ϵ (at least) converges strongly, while in Case (2) the whole sequence converges strongly. We can conjecture that, in fact, the whole sequence should converge in Case (1) as well, but proving it would be very challenging since it requires a uniqueness result of nonclassical entropy solutions. (See also LeFloch [14].)

Proof. We will apply the general convergence framework established by Schonbek [19]. Based on (2.11) and the uniform estimate (2.5a) derived earlier, we can select a subsequence of u^ϵ converging “in the sense” of the Young measures. To apply [19], we only need to control the entropy dissipation measures associated with the equation (2.1). Let U be a smooth function with (at most) linear growth at infinity

and, more precisely, such that U' and U'' are uniformly bounded on \mathbf{R} . Consider the distribution

$$\Gamma^\epsilon = U(u^\epsilon)_t + F(u^\epsilon)_x,$$

where as usual $F' = U' f'$. With obvious notation consider the decomposition

$$\begin{aligned} \Gamma^\epsilon &= \epsilon (U'(u^\epsilon) u_x^\epsilon)_x - \epsilon U''(u^\epsilon) (u_x^\epsilon)^2 + \delta (U'(u^\epsilon) u_{xx}^\epsilon)_x - \delta U''(u^\epsilon) u_x^\epsilon u_{xx}^\epsilon \\ &= \Gamma_1^\epsilon + \Gamma_2^\epsilon + \Gamma_3^\epsilon + \Gamma_4^\epsilon. \end{aligned}$$

The estimates below hold for all smooth function $\theta : \mathbf{R} \times \mathbf{R}_+ \rightarrow \mathbf{R}$ with compact support in (x, t) .

Consider first the term Γ_1^ϵ . By Cauchy-Schwarz inequality, we get

$$\begin{aligned} \left| \int_0^\infty \int_{\mathbf{R}} \Gamma_1^\epsilon \theta \, dx dt \right| &= \left| \int_0^\infty \int_{\mathbf{R}} \epsilon U'(u^\epsilon) u_x^\epsilon \theta_x \, dx dt \right| \\ &\leq \epsilon C \|u_x^\epsilon\|_{L^1(\mathbf{R}_+, L^2(\mathbf{R}))} \|\theta_x\|_{L^\infty(\mathbf{R}_+, L^2(\mathbf{R}))} \\ &\leq C' \sqrt{\epsilon} \rightarrow 0, \end{aligned} \tag{2.12i}$$

where we used (2.5b). This proves that Γ_1^ϵ converges to zero in the sense of distributions.

Next we simply point out that, by (2.5b) again, the second term Γ_2^ϵ remains uniformly bounded in L^1 :

$$\int_0^\infty \int_{\mathbf{R}} |\Gamma_2^\epsilon| \, dx dt \leq \frac{1}{2} \|u_0^\epsilon\|_{L^2(\mathbf{R})}^2. \tag{2.12ii}$$

To estimate Γ_3 we use (2.5d):

$$\begin{aligned} \left| \int_0^\infty \int_{\mathbf{R}} \Gamma_3^\epsilon \theta \, dx dt \right| &= \left| \int_0^\infty \int_{\mathbf{R}} \delta U'(u^\epsilon) u_{xx}^\epsilon \theta_x \, dx dt \right| \\ &\leq \delta C \|u_{xx}^\epsilon\|_{L^1(\mathbf{R}_+, L^2(\mathbf{R}))} \|\theta_x\|_{L^\infty(\mathbf{R}_+, L^2(\mathbf{R}))} \\ &\leq C' \sqrt{\frac{\delta}{\epsilon}} \rightarrow 0, \end{aligned} \tag{2.12iii}$$

provided that the mild condition $\delta = o(\epsilon)$ holds. Therefore Γ_3^ϵ tends to zero in the sense of distributions.

Finally, we deal with the last term as follows:

$$\begin{aligned} \left| \int_0^\infty \int_{\mathbf{R}} \Gamma_4^\epsilon \theta \, dx dt \right| &= \left| \int_0^\infty \int_{\mathbf{R}} \delta U''(u^\epsilon) u_x^\epsilon u_{xx}^\epsilon \theta \, dx dt \right| \\ &\leq \delta C \|u_{xx}^\epsilon\|_{L^\infty(\mathbf{R}_+, L^2(\mathbf{R}))} \|u_x^\epsilon\|_{L^\infty(\mathbf{R}_+, L^2(\mathbf{R}))} \|\theta\|_{L^\infty(\mathbf{R} \times \mathbf{R}_+)} \\ &\leq C' \frac{\sqrt{\delta}}{\epsilon}, \end{aligned} \tag{2.12iv}$$

where we use (2.5b) and (2.5d). The upper bound above tends to zero iff $\delta = o(\epsilon^2)$, in which case we can conclude that Γ_4^ϵ tends to zero in the sense of distributions.

Under the weaker assumption $\delta = O(\epsilon^2)$, we see that Γ_4^ϵ is solely bounded in $L^1(\mathbf{R} \times \mathbf{R}_+)$ as is Γ_2^ϵ .

The conclusion (1) of the theorem follows immediately from the uniform bounds (2.12) by applying Schonbek's convergence theory. Her arguments only show that a subsequence of u^ϵ converges and that the limit is a weak solution of (2.3)-(2.4). On the other hand, assuming now the stronger condition $\delta = o(\epsilon^2)$ and restricting attention to *convex functions* U , in view of (2.12) again and the expression of Γ_2^ϵ we see that the entropy dissipation decomposes in the form

$$\Gamma^\epsilon = \tilde{\Gamma}^\epsilon + \Gamma_2^\epsilon,$$

where $\tilde{\Gamma}^\epsilon \rightarrow 0$ in the sense of distributions and Γ^ϵ is a non-positive bounded measure. This shows that all of the entropy inequalities hold in the limit $\epsilon \rightarrow 0$. Thus the limit coincides with the unique entropy solution of the problem. \square

3. MULTI-DIMENSIONAL CONSERVATION LAWS

The estimates and the technique of proof in Section 2 do not apply to multi-dimensional equations, and markedly different arguments are discussed now. Consider the following Cauchy problem:

$$u_t + \sum_{j=1}^d f_j(u)_{x_j} = \epsilon \sum_{j=1}^d u_{x_j x_j} + \delta \sum_{j=1}^d u_{x_j x_j x_j}, \quad u = u^\epsilon(x, t), \quad x \in \mathbf{R}^d, \quad t > 0, \quad (3.1)$$

$$u(x, 0) = u_0^\epsilon(x), \quad x \in \mathbf{R}^d. \quad (3.2)$$

Provided the initial data u_0^ϵ converge weakly to some limit u_0 (in L^2 , say), we will now prove that the solutions of (3.1)-(3.2) converge toward the entropy solution of the associated hyperbolic problem:

$$u_t + \sum_{j=1}^d f_j(u)_{x_j} = 0, \quad u = u(x, t), \quad x \in \mathbf{R}^d, \quad t > 0, \quad (3.3)$$

$$u(x, 0) = u_0(x), \quad x \in \mathbf{R}^d. \quad (3.4)$$

Precisely our result are as follows:

Theorem 3.1. *Suppose that the flux-function f is Lipschitz continuous on \mathbf{R} and that the initial data u_0 belong to $L^2(\mathbf{R}^d)$. Then the solution u^ϵ of (3.1)-(3.2) satisfies the following a priori estimates:*

$$\|u^\epsilon(t)\|_{L^2(\mathbf{R}^d)} \leq \|u_0^\epsilon\|_{L^2(\mathbf{R}^d)}, \quad t \geq 0, \quad (3.5a)$$

$$\sqrt{2\epsilon} \|u_x^\epsilon\|_{L^1(\mathbf{R}_+, L^2(\mathbf{R}^d))} \leq \|u_0^\epsilon\|_{L^2(\mathbf{R}^d)}, \quad (3.5b)$$

for all $j = 1, \dots, d$ and $t \geq 0$

$$\epsilon \|u_{x_j}^\epsilon(t)\|_{L^2(\mathbf{R}^d)} \leq \sqrt{d} \|f_j'\|_\infty \|u_0^\epsilon\|_{L^2(\mathbf{R}^d)} + \epsilon \|u_{0x_j}^\epsilon\|_{L^2(\mathbf{R}^d)} \quad (3.5c)$$

and for all $j, k = 1, \dots, d$

$$\epsilon^{3/2} \|u_{x_j x_k}^\epsilon\|_{L^1(\mathbf{R}_+, L^2(\mathbf{R}^d))} \leq \sqrt{d} \|f_j'\|_\infty \|u_0^\epsilon\|_{L^2(\mathbf{R}^d)} + \epsilon \|u_{0x_j}^\epsilon\|_{L^2(\mathbf{R}^d)}. \quad (3.5d)$$

For each $u_0 \in L^2(\mathbf{R}^d)$, the Cauchy problem (3.3)-(3.4) admits a unique entropy solution $u \in L^\infty(\mathbf{R}_+, L^2(\mathbf{R}^d))$ in the sense of Kruzkov. See again [10, 4, 20, 9, 14].

Theorem 3.2. Assume that, for some constant $C_0 > 0$ independent of ϵ ,

$$\|u_0^\epsilon\|_{L^2(\mathbf{R}^d)} + \epsilon \sum_{j=1}^d \|u_{0x_j}^\epsilon\|_{L^2(\mathbf{R})} \leq C_0. \quad (3.6)$$

Then, when $\epsilon \rightarrow 0+$ with $\delta = o(\epsilon^2)$, the solution u^ϵ of (3.1)-(3.2) converges in $L_{loc}^p(\mathbf{R}_+, L_{loc}^q(\mathbf{R}^d))$ (for all $1 < p < \infty$ and $1 < q < 2$) toward the unique entropy solution in the sense of Kruzkov of the Cauchy problem (3.3)-(3.4).

Recall again that the condition $\delta = o(\epsilon^2)$ is sharp since, in the opposite case, nonclassical solutions violating the Kruzkov entropy inequalities could arise in the limit.

Proof of Theorem 3.1. We omit the upper-index ϵ in the following calculation. To derive the L^2 bound (3.5a), we multiply the equation (3.1) by u and get

$$\begin{aligned} & \left(\frac{|u|^2}{2}\right)_t + \sum_{j=1}^d F_j(u)_{x_j} \\ &= \sum_{j=1}^d (\epsilon u u_{x_j})_{x_j} - \epsilon \sum_{j=1}^d |u_{x_j}|^2 - \frac{\delta}{2} \sum_{j=1}^d (|u_{x_j}|^2)_{x_j} + \sum_{j=1}^d (\delta u u_{x_j x_j})_{x_j}, \end{aligned}$$

where $F_j' = u f_j'$ is normalized by the condition $F_j(0) = 0$, $j = 1, \dots, d$. Integrating over space, we get

$$\frac{d}{dt} \int_{\mathbf{R}^d} |u|^2 dx = -2\epsilon \int_{\mathbf{R}^d} \sum_{j=1}^d |u_{x_j}|^2 dx$$

and so for all $t \geq 0$

$$\int_{\mathbf{R}^d} |u(t)|^2 dx + 2\epsilon \int_0^t \int_{\mathbf{R}^d} \sum_{j=1}^d |u_{x_j}|^2 dx dt = \int_{\mathbf{R}^d} |u_0|^2 dx. \quad (3.7)$$

To estimate the gradient of u , for $k = 1, \dots, d$ we differentiate the equation (3.1) with respect to the variable x_k and then multiply by u_{x_k} . The right-hand side of (3.1) is linear in u thus the calculation for this side is identical to the one we just made, but with u replaced with u_{x_k} . On the other hand, the flux term in the left-hand side is nonlinear and requires a specific argument:

$$\frac{d}{dt} \int_{\mathbf{R}^d} |u_{x_k}|^2 dx - \sum_{j=1}^d \int_{\mathbf{R}^d} 2 u_{x_k x_j} f_j'(u) u_{x_k} dx = -2\epsilon \sum_{j=1}^d \int_{\mathbf{R}^d} |u_{x_j x_k}|^2 dx,$$

so after integration in time

$$\begin{aligned} & \int_{\mathbf{R}^d} |u_{x_k}(t)|^2 dx + 2\epsilon \sum_{j=1}^d \int_0^t \int_{\mathbf{R}^d} |u_{x_j x_k}|^2 dx dt \\ & \leq \int_{\mathbf{R}^d} |u_{0x_k}|^2 dx + 2 \|f_k'\|_\infty \sum_{j=1}^d \int_0^t \int_{\mathbf{R}^d} |u_{x_j x_k}| |u_{x_k}| dx dt \\ & \leq \int_{\mathbf{R}^d} |u_{0x_k}|^2 dx + \frac{\|f_k'\|_\infty^2}{\epsilon} d \sum_{j=1}^d \int_0^t \int_{\mathbf{R}^d} |u_{x_k}|^2 dx dt + \epsilon \sum_{j=1}^d \int_0^t \int_{\mathbf{R}^d} |u_{x_j x_k}|^2 dx dt. \end{aligned}$$

Observe that the last term of the right-hand side coincides with the last term of the left-hand side. Therefore, multiplying the above inequality by ϵ^2 and using the entropy dissipation bound in (3.7), we deduce that

$$\begin{aligned}
& \int_{\mathbf{R}^d} \epsilon^2 |u_{x_k}(t)|^2 dx + \sum_{j=1}^d \int_0^t \int_{\mathbf{R}^d} \epsilon^3 |u_{x_j x_k}|^2 dx dt \\
& \leq \int_{\mathbf{R}^d} \epsilon^2 |u_{0x_k}|^2 dx + \|f'_k\|_\infty^2 \int_0^t \int_{\mathbf{R}^d} d \epsilon |u_{x_k}|^2 dx dt \\
& \leq \int_{\mathbf{R}^d} \epsilon^2 |u_{0x_k}|^2 dx + d \|f'_k\|_\infty^2 \int_{\mathbf{R}^d} |u_0|^2 dx.
\end{aligned} \tag{3.8}$$

□

Proof of Theorem 3.2. We will rely on the convergence framework proposed by DiPerna [4] for L^∞ solutions and generalized to L^p solutions by Szepessy [20] and Kondo and LeFloch in [9].

Consider a Young measure ν associated with the sequence u^ϵ and based on the uniform L^2 bound (3.5a). (Such Young measures are described in Schonbek [19]). To show that ν is an entropy measure-valued solution, we must check entropy inequalities associated with the equation (3.3), that is,

$$\langle \nu, U \rangle_t + \sum_{j=1}^d \langle \nu, F_j \rangle_{x_j} \leq 0, \tag{3.9}$$

where $U : \mathbf{R} \rightarrow \mathbf{R}$ is a convex function with (at most) linear growth at infinity and the entropy flux $F'_j = U' f'_j$ is normalized so that $F_j(0) = 0$.

By the definition of the Young measure, we only need to establish that, in the decomposition

$$\begin{aligned}
& \partial_t U(u^\epsilon) + \sum_{j=1}^d \partial_j F_j(u^\epsilon) \\
& = \sum_{j=1}^d \partial_j (\epsilon U'(u^\epsilon) \partial_j u^\epsilon + \delta(\epsilon) U'(u^\epsilon) \partial_j^2 u^\epsilon) \\
& \quad - \sum_{j=1}^d \epsilon U''(u^\epsilon) |\partial_j u^\epsilon|^2 + \delta(\epsilon) U''(u^\epsilon) \partial_j u^\epsilon \partial_j^2 u^\epsilon \\
& =: \Gamma_1^\epsilon + \Gamma_2^\epsilon + \Gamma_3^\epsilon + \Gamma_4^\epsilon,
\end{aligned} \tag{4.11}$$

we have

$$\Gamma_1^\epsilon, \Gamma_2^\epsilon, \Gamma_4^\epsilon \rightarrow 0$$

and

$$\Gamma_3^\epsilon \leq 0.$$

These convergence properties precisely were established in the proof of Theorem 2.2, at least for one-dimensional equations. The extension to multi-dimensional

equations is immediate in view of the uniform estimates (3.5). A detailed discussion of the initial condition at $t = 0$ (which is based on using suitable entropy inequalities) can be found in Kondo and LeFloch in [9]. This completes the proof that the convergence framework in [9] applies and provides the strong convergence toward the unique entropy solution of (3.3)-(3.4). \square

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